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STOCHASTIC CONTROL OF TWO COMPETING QUEUES

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ABSTRACT

We consider optimal server time allocation to two parallel queues. The server has available complete past observations of the queue sizes for his decisions. The infinite time discounted version of the problem is analyzed here. It is shown that the optimal strategy is stationary. The optimal value function is shown to be the unique solution of the Bellman equation. Finally, analysis of degenerate Bellman equations, of the type appearing in this problem is presented. Numerical methods of solution can be derived from the results presented here.

1. INTRODUCTION

We consider the problem of selecting which of two parallel queues to serve with a single server. The system is depicted in Figure 1. Customers arrive into stations 1 and 2 according to two independent Poisson streams with constant rates $\lambda_1$, $\lambda_2$ respectively. The two queues compete for the services of an exponential server with constant service rate $\mu$. Let $x_{1,t}$ be the number of customers in queue 1 at time $t$, the customer in service included. The control to be selected is clearly of a switching type. When $u_t=1$ and the server completes a service the next customer to be served comes from queue 1, while if $u_t=0$ the next customer comes from queue 2. The queue that is being served forms an $M/M/1$ system with the server. This is a simple sequencing problem, where the sequencing variable is $u_t$. Let $x_t=(x_{1,t},x_{2,t})$ denote the state at time $t$. $u_t$ is to be selected knowing $x_t$.

The server allocation time is to be selected to minimize delays, weighted according to $c_1$, $c_2$, two positive constants. Thus the cost per unit time with queues $x_{1,t}$, $x_{2,t}$ is $c_1x_{1,t}+c_2x_{2,t}=c^T x_t$. It is shown that the policies which minimize the infinite time discounted average cost are characterized by a switching curve $x_2<^S(x_1)$. That is $u_t=1$ if $x_{2,t}<^S(x_{1,t})$, while $u_t=0$ when $x_t<^S(x_{1,t})$. Thus the optimal strategy is stationary.

The present paper is an outgrowth of earlier work [1] by one of the authors. Research supported in part by the US Department of Energy under contract DE-AC01-79ET 29244.

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There are two important differences between the work presented in [1] and the problems treated here. First the queues are observed here while only partial observations were assumed in [1]. Second finite time problems were treated in [1] while infinite time discounted problems are considered here.

The problem described can be thought of as a dynamic priority (non-preemptive) problem in a single queue with two classes of customers [2] [3]. The optimal static priority assignment (i.e. open loop control) which minimizes average cost per unit time, under steady state conditions was obtained by Cox and Smith [2]. The result, in the terminology used here, is that queue 1 has priority over queue 2 iff \( c_1 > c_2 \) (the opposite holds when \( c_2 < c_1 \)). Actually their result is easily extended to include general service time distributions, as well as different distributions for different classes and to general \( m \) queues (classes) problems. Dynamic priorities minimizing average cost per unit time, under steady state conditions were derived by Rykov and Lembert [3]. The analysis presented in [3] is rather formal, however the results appear to be correct. The result, in the terminology used here, is that the optimal dynamic (feedback) priority coincides with the optimal static priority of Cox and Smith [2]. The result is valid under general service time distributions. In both cases the results were obtained by classical queuing methods and are in agreement with intuition, given the steady state assumptions and the average cost per unit time criterion used.

The problem considered here is quite different from those analyzed in [2] [3], since infinite time discounted average delays are considered. We use two methods to analyze the problem. The first uses an embedded Markov chain and is inspired by the analysis of a simple tandem queuing system studied by Rosberg, Vaynaya and Walrand [6]. The second centers on direct analysis of the associated Bellman equation. Both methods suggest numerical schemes for computation of the optimal strategy, and are useful for more general queuing systems problems as well.

Other examples of dynamic queuing control problems can be found in [4] [5].

2. A METHOD BASED ON THE ASSOCIATED EMBEDDED MARKOV CHAIN PROBLEM

We consider the discounted continuous time problem for the two competing queues system described in figure 1. Let \( \alpha > 0 \) be the discount rate and

\[
\nu_t^\alpha(x) = \min_{\gamma \in \Gamma} \mathbb{E} \left[ \int_t^\infty c_x \gamma e^{-\alpha \sigma} d\sigma \right] \tag{2.1}
\]

be the minimum total expected discounted delay, when the time horizon is \( \tau > 0 \) and the starting state at time 0 is \( x_0 = x \). In (2.1) \( \Gamma \) is the set of all admissible policies

\[
\Gamma = \{ \gamma^\tau | u^\tau(\cdot), \tau \geq 0 \} \text{ such that } u^\tau : \mathbb{Z}_+ \times \{1,0\}
\]

where \( \mathbb{Z} = \{0,1,2,3,\ldots\} \).

The main objective is to characterize the optimal cost for the infinite time problem \( \nu_\infty^\alpha \) and the corresponding optimal strategy \( \gamma_\infty^\alpha \). Considering the sizes of both queues \( x_t \), as the state of the system, we have a Markov chain with countable state space \( \mathbb{Z}_+ \times \mathbb{Z} \). The transitions of \( x_t \) are easy to describe. They correspond to arrivals to queue 1 or queue 2, and to service completions. The transitions induced by arrivals occur at arrival times in queues 1 and 2 respectively denoted by.
\[ u^1 = (t^1_0, t^1_1, t^1_2, \ldots) \]
\[ u^2 = (t^2_0, t^2_1, t^2_2, \ldots) \]

The control strategy does not affect arrivals. Therefore if we let \( A_1, A_2 \)
be functions describing the new state after a transition induced by arrivals
we see that for \( u = 1 \) or \( 0 \), the transition probabilities are

\[
\begin{align*}
A_1(x_1, x_2) &= (x_1 + t^1_1, x_2) \quad \text{with Prob. } \lambda_1 dt + o(dt) \\
A_2(x_1, x_2) &= (x_1, x_2 + t^2_1) \quad \text{with Prob. } \lambda_2 dt + o(dt)
\end{align*}
\]

On the other hand transitions induced by service completions occur at
service completion times

\[
J = (t^1_0, t^1_1, t^1_2, \ldots)
\]

and depend on the value of the control. Thus when \( u = 1 \), \( D_1 \) is possible where

\[
D_1(x_1, x_2) = ((x_1 - 1)^+, x_2) \quad \text{with Prob. } ud t + o(dt),
\]

while when \( u = 0 \), \( D_2 \) is possible where

\[
D_2(x_1, x_2) = (x_1, (x_2 - 1)^+) \quad \text{with Prob. } ud t + o(dt)
\]

Here \((x)^+ = \max\{x, 0\}\). The transition epochs \(2, 3, 6\), for the state
process \(x_t\) are

\[
J = u^1 \cup u^2 \cup J
\]

The embedded Markov chain method analyzes the behavior of the state process
\(x_t\) at transition epochs from \(J\) \(2, 3, 6\). It is a consequence of our
formulation that the sets \(u^1, u^2, J\) are disjoint (i.e. no simultaneous
occurrence is possible).

To treat the infinite time discounted problem we need some results of
Lippman \(8\), particularly since we do not have bounded costs for all \( x\). The
action space in the problem treated here is finite: \(0, 1\). The set of states
accessible from a given state \((x_1, x_2)\) in one transition is \(\{(x_1 + t^1_1, x_2),
(x_1, x_2 + t^2_1), (x_1 - 1)^+, x_2), (x_1, (x_2 - 1)^+), (x_1 x_2)\}\), independent of the transition epoch. Furthermore the cost is linear in the state. As a consequence
Assumptions 1 and 2 in Lippman \(8, \text{pp. 719}\) hold.

The strategies in \(T\) are Markovian \(9\). Let us also consider the more
general class of randomized nonanticipative strategies:

\[
\gamma_t = \{\gamma_t(x_t(0), x_t(1)), t \geq 0\} \quad \text{such that } \gamma_t \text{ is a function of the past history } h^t(x_0, u_0, \text{0} \leq s \leq t) \}
\]

Here \(\gamma_t(1), i = 0, 1\) is the probability of choosing \(u = 0\), or \(1\) at time \(t\). A
strategy in \(T\) is stationary whenever \(u_t(\cdot)\) is independent of \(t\). We now have
the following result:

**Theorem 1:**

The infinite time discounted problem \((a > 0)\), for the two competing queues,
has an optimal strategy over \(T\), which is Markovian and stationary. Furthermore
the optimal value function \(V^a\) is the unique solution of the Bellman
\[
V^d_{i_1, i_2} = \frac{c_1}{\mu + \alpha} I_1 + \frac{c_2}{\mu + \alpha} I_2 + \min_{v \epsilon \{0, 1\}} \left( \sum_{i_1, i_2} P^1_{i_1, i_1} (v) P^2_{i_2, i_2} (v) V^d_{v(i_1, i_2)} \right)
\]
(2.10)

\[i_1, i_2 \geq 0,
\]

where \(P^1_{i_1, i_1} (v), P^2_{i_2, i_2} (v)\) are given in (2.21) below.

**Proof:** Since Assumptions 1, 2 of Lippman [8] are satisfied and the action set is finite, the first result is an immediate consequence of Theorem 1 in Lippman [8, p. 719]. Since the optimal policy is stationary we can restrict consideration to stationary Markovian strategies which change values only at service completion times, \(t_k^s \epsilon J\). We denote this restricted class by \(\Gamma_0\). As a result we need find the equivalent discrete time stochastic control problem for the embedded Markov chain with transition epochs \(J\) and not \(J\). For ease of notation we denote \(x_k\) by \(x_k\) and \(u_k\) by \(u_k\). It is clear that the intertransition intervals \(t_{k+1}^s - t_k^s\) are independent, identically distributed random variables with an exponential distribution.

\[
\text{Prob} \{t_{k+1}^s - t_k^s > t\} = \exp(-\alpha t).
\]
(2.11)

Consider now a policy in \(\Gamma_0\) and let us compute the corresponding cost over the random interval \([0, t^s]\) and initial state \(x\):

\[
E_x \int_0^{t^s} e^{-\alpha t} c^T x_t dt = E_x \sum_{k=0}^{n-1} \int_{t_k^s}^{t_{k+1}^s} e^{-\alpha t} c^T x_k dt
\]

\[
= E_x \sum_{k=0}^{n-1} E_x \underbrace{e^{-\alpha t} c^T x_k}_{t_k^s}
\]

\[
= E_x \sum_{k=0}^{n-1} \left( c^T x_k \frac{E_x e^{-\alpha t}}{E_x e^{-\alpha t}} \right)
\]

\[
= E_x \sum_{k=0}^{n-1} \left( -\frac{c^T x_k}{\alpha} \right)
\]

\[
= \frac{1}{\alpha} E_x \sum_{k=0}^{n-1} c^T x_k (\beta - 1) + \frac{1-\beta}{\alpha} E_x \sum_{k=0}^{n-1} c^T x_k
\]

(2.12)

provided \(\beta < 1\), while when \(\beta = 1\) it equals \(E_x \sum_{k=0}^{n-1} c^T x_k\). In the sequel we shall ignore the constant \(\frac{1-\beta}{\alpha}\) in (2.12). Here

\[
\beta = \frac{\mu}{\mu + \alpha},
\]

so \(\beta = 1\), corresponds to the discount rate \(\alpha = 0\). So the infinite time cost for the embedded Markov chain is

\[
\gamma^d_0(y, x) = E_x \sum_{k=0}^{\infty} \beta^k c^T x_k
\]

(2.14)

when policy \(\gamma^d_0\) is used. Obviously for \(\alpha > 0\),

\[
\gamma^d_0 = \frac{1-\beta}{\alpha} \gamma^d
\]

(2.15)
where \( \mathcal{W} \) is the optimal over \( \gamma \mathcal{R} \), of (2.14). We next compute the transition probabilities for the embedded Markov chain \( x \). Let \( \xi_{1,k}, \xi_{2,k} \) be the total arrivals for queues 1, 2 during the interval \([t_k, t_{k+1})\]. Note that given \( t_{k+1} - t_k = \delta, \xi_{1,k}, \xi_{2,k} \) are independent random variables, with Poisson distributions, with means \( \lambda_1 \delta, \lambda_2 \delta \) respectively. Let \( N_{1,t}, N_{2,t} \) be the counting processes for queues 1, 2.

Then
\[
\Pr(N_{1,k+1} - N_{1,k} = \xi_{1,k}) = \int_0^\infty \frac{1}{(\lambda_1 \delta)} e^{-\lambda_1 \delta} \xi_{1,k} \cdot \frac{\mu}{\lambda_1 + \mu} \, d\delta
\]
(2.16)

Similarly
\[
\Pr(N_{2,k+1} - N_{2,k} = \xi_{2,k}) = \frac{\lambda_2}{\lambda_2 + \mu} \xi_{2,k} \cdot \frac{\mu}{\lambda_2 + \mu}
\]
(2.17)

Now if \( u_k = 1 \)
\[
\begin{align*}
x_{1,k+1} &= x_{1,k} + 1 + \xi_{1,k}, \quad \text{if } x_{1,k} \neq 0 \\
x_{1,k+1} &= \xi_{1,k}, \quad \text{if } x_{1,k} = 0 \\
x_{2,k+1} &= x_{2,k} + \xi_{2,k}
\end{align*}
\]
(2.18)

Similarly if \( u_k = 0 \)
\[
\begin{align*}
x_{1,k+1} &= x_{1,k} + \xi_{1,k} \\
x_{2,k+1} &= x_{2,k} + 1 + \xi_{2,k}, \quad \text{if } x_{2,k} \neq 0 \\
x_{2,k+1} &= \xi_{2,k}, \quad \text{if } x_{2,k} = 0
\end{align*}
\]
(2.19)

So the (reduced) embedded Markov chain has state space \( \mathbb{Z}x\mathbb{Z} \) and transition probabilities
\[
P_{1_j,2_j,1_j,2_j}(u) = \Pr(x_{k+1}=(j_1,j_2)|x_k=(l_1,l_2), u_k=v)
\]
\[
= \Pr(x_{1,k+1}=j_1|l_1, u_k=v) \cdot \Pr(x_{2,k+1}=j_2|l_2, u_k=v)
\]
(2.20)

From (2.16) - (2.19) there result
\[
P_{1_j,1_j}(1) = \begin{cases} 
(\lambda_1/\lambda_1 + \mu)^{j_1}(u/\lambda_1 + \mu), & \text{if } j_1 = 0, j_2 \geq 0 \\
(\lambda_1/\lambda_1 + \mu)^{j_1+1-1}(u/\lambda_1 + \mu), & \text{if } j_1 \geq 1, j_2 \geq j_1 - 1 \\
0, & \text{otherwise}
\end{cases}
\]
(2.21a)

\[
P_{1_j,1_j}(0) = \begin{cases} 
(\lambda_1/\lambda_1 + \mu)^{j_1-1}(u/\lambda_1 + \mu), & \text{if } j_1 \geq j_2 \\
0, & \text{otherwise}
\end{cases}
\]
(2.21b)
$$p_{12}^{2}(j_2) = \begin{cases} \frac{\lambda_2}{\lambda_2 + \mu} j_2 \lambda_2^{-1/2} (\mu/\lambda_2 + u), & \text{if } j_2 > 0 \\ 0, & \text{otherwise} \end{cases} \quad (2.21c)$$

$$p_{12}^{2}(0) = \begin{cases} \frac{\lambda_2}{\lambda_2 + \mu} j_2^{1/2} (\mu/\lambda_2 + u), & \text{if } j_2 > 0 \\ 0, & \text{otherwise} \end{cases} \quad (2.21d)$$

Then $\psi_\alpha$ is the unique solution of

$$w_\alpha(1, 1, 2) = \psi_\alpha(1, 1, 2) = c_1 + c_2 1 + 2 \min_{v \in \{0, 1\}} \psi_\alpha(1, 1, 2) \quad (2.22)$$

Multiplying (2.22) by $\frac{1 - \beta}{\alpha}$ we obtain (2.10), and this completes the proof of the theorem.

It is now clear how to obtain the switching curve in $Z \otimes Z$, once $\psi_\alpha$ is known:

$$S = \{(1, 1, 2), \in Z \otimes Z:$$

$$\psi_\alpha(1, 1, 2) = \psi_\alpha(1, 1, 2)$$

From (2.21) easily follows that $(0, 0) \in S$. The optimal stationary policy $v_\alpha = (v^g, v^f, v^a, \ldots)$ is determined by the function $f^g: Z \otimes Z \to (0, 1)$ as follows. If $(1, 1, 2)$ is such that

$$\mathbb{E} \left[ p_1^{1}(1) p_2^{2} (1) \psi_\alpha(1, 1, 2) \right] = \mathbb{E} \left[ p_1^{1}(0) p_2^{2} (0) \psi_\alpha(1, 1, 2) \right]$$

then

$$f^g(1, 1, 2) = 1 \quad (2.24a)$$

and

$$f^g(1, 1, 2) = 0 \quad (2.24c)$$

when the inequality in (2.24) is reversed. From (2.21), (2.24) easily follows that $f^g(0, 1, 2) = 0$ and $f^g(1, 0, 1) = 1$. These results agree with intuition.

The result of Theorem 1, suggests the following scheme for the numerical computation of $\psi_\alpha$ and $f^g$.

Let

$$\psi_\alpha(1, 1, 2) = \frac{c_1}{\mu + c_2} 1^2 + \frac{c_2}{\mu + c_2} 1_2, \quad 1, 1_2 \geq 0 \quad (2.25)$$

and define $\psi_\alpha$ recursively via
\[
\nu_n^a(i_1, i_2) = \frac{c_1}{\mu + a} i_1 + \frac{c_2}{\mu + a} i_2 + \frac{\mu}{\mu + a} \min_{v \in (0,1)} \left\{ \sum_{j_1, j_2} p_{i_1, i_2}^{j_1, j_2} (v) p_{j_1, j_2}^{i_1, i_2} \right\} v_n^{a-1}(j_1, j_2)
\]

(2.26)

Let

\[
f_n^a, Z \in \mathcal{Z} \in (0,1)
\]

be the function defined by the minimization in (2.26). Then we can show that

\[
\lim_{n \to \infty} \nu_n^a = \nu^a
\]

(2.27)

3. DIRECT ANALYSIS OF THE BELLMAN EQUATION

First recall from [10] the following result on Bellman equations over an arbitrary Hilbert lattice \( V \):

\[
\max_{a \in A} (L^a u - f^a) = 0, \quad ucV, f^a \in V
\]

(3.1)

where \( L^a \) is coercive, and \( A \) is the action set.

Let

\[
K = \{ ucV : L^a u \leq f^a, \quad \forall a \in A \}
\]

(3.2)

Next for arbitrary \( a \in A \), let \( \tilde{\psi} \) be the solution of the variational inequality

\[
< L^a \tilde{\psi} - f^a, v - \tilde{\psi} > \geq 0 \quad \forall v \in \mathcal{K}, v \in \mathcal{K}
\]

(3.3)

Then (see [10]) \( \tilde{\psi} \) is the maximal element of the set of all subsolutions of (3.1) (which implies that \( \tilde{\psi} \) is independent of the choice of \( a \) in (3.2)). Furthermore, under appropriate technical assumptions [10], \( \tilde{\psi} \) is also the strong solution of (3.1). In view of the result of Theorem 1 in section 2, it is only necessary to consider stationary problems.

For the case of interest here \( A = \{1, 0\} \), i.e. we have two operators \( L_1, L_0 \). Furthermore we have two state variables \( x_1, x_2 \) which are integer valued. Let us define

\[
f(i_1, i_2) = \frac{c_1}{\mu + a} i_1 + \frac{c_2}{\mu + a} i_2, \quad i_1, i_2 \geq 0
\]

(3.4)

Furthermore for a function \( v(i_1, i_2) \) let

\[
[L^0 v](i_1, i_2) = v(i_1, i_2) - \frac{\mu}{\mu + a} \sum_{j_1, j_2} p_{i_1, i_2}^{j_1, j_2} (0) p_{j_1, j_2}^{i_1, i_2} (0) v(j_1, j_2)
\]

\[
[L^1 v](i_1, i_2) = v(i_1, i_2) - \frac{\mu}{\mu + a} \sum_{j_1, j_2} p_{i_1, i_2}^{j_1, j_2} (1) p_{j_1, j_2}^{i_1, i_2} (1) v(j_1, j_2)
\]

(3.5)

Then (2.10) can be written

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\[
\max \left( L_0^0v, L_0^1v \right) = f
\] (3.6)

The main difficulty arising here is due to the fact that \( L_0, L_1 \) are not coercive, but degenerate. Thus, it is necessary to modify the method of [10] to apply to degenerate problems; this is done in [11]; here we present the method for the particular case of two operators.

It is easier to explain the main features of this method by introducing some functional spaces. Because our arguments do not depend on the maximum principle, they are applicable to both, the continuous and the discrete problems.

Let \((1, 2, \ldots, n) = \mathcal{A} \cup \mathcal{B}\), where \( n \) is the dimension of the spatial variable and \( \mathcal{A} \cup \mathcal{B} \neq \emptyset \). Let \( \mathcal{A} \subseteq \mathbb{R}^a \), \( \mathcal{B} \subseteq \mathbb{R}^b \), where \( \mathcal{A} \subseteq \mathbb{R}^a \) and \( \mathcal{B} \subseteq \mathbb{R}^b \) are bounded open sets in the respective spaces. (The case of an unbounded set \( \mathcal{A} \) can be handled similarly, by introducing weighted functional spaces).

Let
\[
L_1 \equiv L_{1,1} + L_{1,2}, \quad L_0 \equiv L_{0,1} + L_{0,2}
\]

where \( L_{1,1} \) is a first-order hyperbolic (differential or finite difference) operator in the \( A \)-variables and \( L_{1,2} \) is coercive in the \( B \)-variables, while the situation is reversed for \( L_{0,1,2} \).

Now let
\[
\begin{align*}
W_A &= H^1_0(\Omega_A), \quad W_B = H^1_0(\Omega_B) \\
\mathcal{D}_A &= \{ \text{vel} L^2(\Omega); \ \nu \vert_{\partial L_{1,1} \cap \Omega} \in L^2(\partial \Omega_A), \ \nu = 0 \text{ on } \Gamma_A \}
\end{align*}
\]

where
\[
\Gamma_A = \{ x \in \partial \Omega_A : \langle L_{1,1} \nu \rangle < 0 \}.
\]
\( \mathcal{D}_B \) is defined similarly.

(Note: \( \mathcal{D}_A \) is defined similarly. \\
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\( \mathcal{D}_B \) is defined similarly. (Note: \( \mathcal{D}_B \) is defined similarly.)

Finite differences must be substituted instead of derivatives for the discrete problem. Further we denote by \( H_A, H_B \) the spaces \( L^2(\mathcal{G}_A), L^2(\mathcal{G}_B) \) (or \( \ell^2(\mathcal{G}_A), \ell^2(\mathcal{G}_B) \), for finite difference equations). We define the spaces
\[
V_A = \mathcal{D}_A \otimes W_B, \quad V_0 = W_A \otimes \mathcal{D}_B; \ U_1 = H_A \otimes W_B, \ U_0 = W_A \otimes H_B.
\]

(Note: \( \otimes \) signifies tensor product.)

Let the convex set \( K \) be defined by
\[
K = \{ \text{vel} V_A \otimes V_0; \ L_1 \text{vel} f_1, \ L_0 \text{vel} f_0 \}.
\]

We observe that \( K \) is sup-stable, i.e., \( u, w \in K \) implies \( u + w \in K \).

Let \( \overline{K} \) be the closure of \( K \) in \( U_1 \), and \( \overline{K} = \overline{K} \cap V_1 \).

Let \( u \) be the solution of the variational inequality
\[ <L_{1,1}v + L_{1,2}u - f_1, v - u> \]
\[ - \frac{1}{2} \int_\Omega (v-u)^2 dx - \frac{1}{2} \int_{\Gamma_A} \nu \cdot n_A \nu v^2 d\sigma \geq 0 \]
for all \( v \in K_n \); \( u \in \overline{K}_1 \) \hfill (3.3)

Then we have the following:

**Theorem 2:**

There exists a solution \( u \) of (3.3) and is equal to the maximal element of \( \overline{K}_1 \).

Existence follows by a penalization argument as in [12]. The claim that \( u \) is the maximal element of \( \overline{K}_1 \) follows by methods similar to those of [10].

When \( f \geq 0 \), we have the following iterative scheme for computing the solution of (2).

Let \( u_0^a = 1, 0 \) be subsolutions of (1), i.e. \( u_0^a \in \mathcal{V}, L u_0^a \leq a \). Suppose that \( u_0^a \in \overline{K}_1 \). Let \( u_n^a \) be defined inductively as the solutions of the variational inequalities

\[ <L_{a_n}^a v - f_n^a, v - u_n^a \geq 0 \] for all \( v \in \mathcal{K}_n, u_n^a \in \mathcal{K}_1 \) \hfill (3.4)

where \( K_n^a = (v \in \mathcal{V}: \nu u_n^a \leq a+1) \),

\[ \overline{K}_n^a = \text{closure of } K_n^a \text{ in } \mathcal{U}_n \],

\[ \mathcal{K}_n^a = \overline{K}_n^a \mathcal{V} \]

\( a = 1, 0. \)

The variational inequalities (3.4) have strong solutions if we make the regularity assumption:

(A) \( u_n^a \in \mathcal{V} \)

Conditions under which the regularity assumption (A) is true are given in [13].

Then, we have the following:

**Theorem 3:**

\( u - u_n \) solution of (3.1), in weak \( \mathcal{U}_n \) and \( u \in C^0(\Omega) \) (\( u \in k^0(\Omega) \), for the finite difference problem).

**References**


